Максвелова расподела брзина и молекулски судари

5.40 $\langle v^n \rangle = ?$

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2kT}}$$

$$\langle v^n \rangle = \int_0^{+\infty} v^n \cdot f(v)\, dv$$

$$\langle v^n \rangle = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^{+\infty} v^{n+2} \cdot e^{-\frac{mv^2}{2kT}} \cdot dv$$

[смена: $\frac{mv^2}{2kT} = t \Rightarrow v = \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \cdot t^{\frac{1}{2}}$

$dv = \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot t^{-\frac{1}{2}} dt$]

$$\langle v^n \rangle = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \int_0^{\infty} \left(\frac{2kT}{m}\right)^{\frac{n+2}{2}} \cdot t^{\frac{n+2}{2}} \cdot e^{-t} \cdot \left(\frac{2kT}{m}\right)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot t^{-\frac{1}{2}} dt =$$

$$= 4 \cdot \pi \cdot \pi^{-\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{-\frac{3}{2}} \cdot \int_0^{\infty} \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \cdot t^{\frac{n+1}{2}} \cdot e^{-t} \cdot \frac{1}{2}\, dt =$$

$$= 4 \cdot \pi^{-\frac{1}{2}} \cdot \left(\frac{2kT}{m}\right)^{-\frac{3}{2}} \cdot \left(\frac{2kT}{m}\right)^{\frac{n+3}{2}} \cdot \frac{1}{2} \cdot \underbrace{\int_0^{\infty} t^{\frac{n+1}{2}} \cdot e^{-t}\, dt}_{} =$$

гама ф-ја: $\int_0^{\infty} x^{n+1} \cdot e^{-x}\, dx = \Gamma(n)$

$$\int_0^{\infty} t^{\frac{n+3}{2}-1} \cdot e^{-t}\, dt = \Gamma\left(\frac{n+3}{2}\right)$$

$$= 4 \cdot \pi^{-\frac{1}{2}} \cdot 2^{-\frac{3}{2}} \cdot \left(\frac{kT}{m}\right)^{-\frac{3}{2}} \cdot 2^{\frac{n+3}{2}} \cdot \left(\frac{kT}{m}\right)^{\frac{n+3}{2}} \cdot 2^{-1} \cdot \Gamma\left(\frac{n+3}{2}\right) =$$

$$= \frac{4}{\sqrt{\pi}} \cdot 2^{\frac{n-2}{2}} \cdot \left(\frac{kT}{m}\right)^{\frac{n+3-3}{2}} \cdot \Gamma\left(\frac{n+3}{2}\right) =$$

$$= \frac{2^2}{\sqrt{\pi}} \cdot 2^{\frac{n-2}{2}} \cdot \left(\frac{kT}{m}\right)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n+3}{2}\right) = \frac{2^{\frac{n+2}{2}}}{\sqrt{\pi}} \cdot \left(\frac{kT}{m}\right)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n+3}{2}\right)$$

$$\boxed{\langle v^n \rangle = \frac{2^{\frac{n+2}{2}}}{\sqrt{\pi}} \cdot \left(\frac{kT}{m}\right)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n+3}{2}\right)}$$

5.41 Највероватнија брзина молекула: $\Rightarrow$ брзина за коју Максвелова расподела брзина има максимум.

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot v^2 \cdot e^{-\frac{mv^2}{2kT}}$$

$\rightarrow$ услов максимума: $\boxed{\frac{df(v)}{dv} = 0}$

$$\frac{df(v)}{dv} = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left[2v \cdot e^{-\frac{mv^2}{2kT}} + v^2 \cdot e^{-\frac{mv^2}{2kT}} \cdot \left(-\frac{\cancel{2}mv}{\cancel{2}kT}\right)\right] =$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left[2v \cdot e^{-\frac{mv^2}{2kT}} - \frac{mv^3}{kT} \cdot e^{-\frac{mv^2}{2kT}}\right] =$$

$$= 4\pi \cdot \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot v \cdot e^{-\frac{mv^2}{kT}} \left[2 - \frac{mv^2}{kT}\right] = 0$$

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$$\Rightarrow v \cdot e^{-\frac{mv^2}{kT}} \left[2 - \frac{mv^2}{kT}\right] = 0 \quad \Rightarrow 2 - \frac{mv^2}{kT} = 0$$

$v \neq 0$

$$\frac{mv^2}{kT} = 2 \quad \Rightarrow v = \sqrt{\frac{2kT}{m}}$$

$$v_{nv} = \sqrt{\frac{2kT}{m}}$$

5.42 Израчунати $\langle v \rangle$ и v_{nv} молекула N_2 на температури 25 °C :

$$\langle v^n \rangle = \frac{2^{\frac{n+2}{2}}}{\sqrt{\pi}} \cdot \left(\frac{kT}{m}\right)^{\frac{n}{2}} \cdot \Gamma\left(\frac{n+3}{2}\right)$$

$$n=1 \Rightarrow \langle v \rangle = \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} \cdot \left(\frac{kT}{m}\right)^{\frac{1}{2}} \cdot \underbrace{\Gamma(2)}_{\Gamma(2)=1} = \left(\frac{2^3 kT}{\pi m}\right)^{\frac{1}{2}} = \sqrt{\frac{8kT}{\pi m}}$$

$k = 1{,}38 \cdot 10^{-23} \frac{J}{K}$

$T = 298{,}15\ K$

$$\sqrt{\frac{J \cdot K}{g}} = \sqrt{\frac{N \cdot m}{g}} = \sqrt{\frac{kg \cdot \frac{m^2}{s^2}}{g}} = \sqrt{\frac{10^3 g \frac{m^2}{s^2}}{g}} = \sqrt{10^3} \cdot \frac{m}{s}$$

$\frac{m}{M} = \frac{N}{N_A}$

$m = 28\ g \cdot \frac{1}{6 \cdot 10^{23}} \approx 4{,}667 \cdot 10^{-23} g = 4{,}667 \cdot 10^{-26} kg$

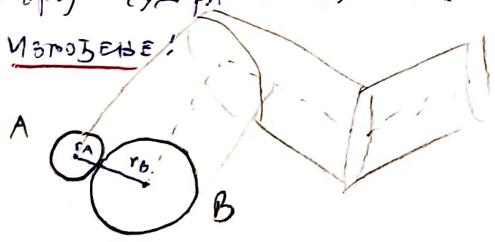
$\langle v \rangle = 475 \frac{m}{s}$ $\quad v_{nv} = \sqrt{\frac{2kT}{m}} = 421 \frac{m}{s}$

5.43 Колико сударa у 1s претрпи молекул N_2 на $p = 10\ atm$ и температури од 25 °C . Површина попречног пресека молекула N_2 је $0{,}43\ nm^2$.

$\langle v \rangle = 475 \frac{m}{s}$

→ број сударa молекула N_2 по јединици времена : $\nu = \sqrt{2} \langle v \rangle \cdot \delta \cdot \frac{p}{kT}$

→ Изводење:



} измишљени цилиндар кроз простор

$r = r_A + r_B$

N_A $\quad n_A = N_B$

→ Задатак (5.43)

молекул N_2 => колико сударa у 1 s притом

$p = 10\ atm$

$T = 25°C \Rightarrow T = 298{,}15\ K$

$b = \delta_{N_2} = 0{,}43\ nm^2$

$\langle v \rangle = 475\ \frac{m}{s}$

→ Број сударa молекулa N_2 по јединици времена износи:

$$\nu = \sqrt{2} \cdot \langle v \rangle \cdot b \cdot \frac{p}{kT}$$

$$\nu = 1{,}41 \cdot 475\ \frac{m}{s} \cdot 0{,}43 \cdot 10^{-18}\ m^2 \cdot \frac{10 \cdot 101\,325\ Pa}{1{,}38 \cdot 10^{-23}\ \frac{J}{K} \cdot 298{,}15\ K}$$

$$\nu = 7{,}1 \cdot 10^{10}\ \frac{1}{s}$$

јединице: $\frac{m}{s} \cdot m^2 \cdot \frac{Pa}{J} = \frac{m^3}{s} \cdot \frac{\frac{N}{m^2}}{N \cdot m} = \frac{m^3}{s} \cdot \frac{1}{m^3} = \frac{1}{s}$

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5.44 Стаклена сијалица $\Rightarrow 10^{20}$ молекула N_2 у $V = 10\,cm^3$ на $25\,^\circ C$

Израчунати број судара једног молекула N_2 по јединици времена.

$\sigma_{N_2} = 0{,}43\ nm^2$ $\quad \nu = \sqrt{2} \cdot \langle v \rangle \cdot \sigma \cdot \frac{N}{V}$

$\nu = 1{,}41 \cdot 475\ \frac{m}{s} \cdot 0{,}43 \cdot 10^{-18}\ m^2 \cdot \frac{10^{20}}{10 \cdot 10^{-6}\ m^3}$

$\nu = 2{,}9 \cdot 10^9\ \frac{1}{s}$

5.45 Лаб. вакуумска пумпа $\Rightarrow$ може да створи $p = 1\ \mu Pa$

Израчунати бр. судара по јединици времена молекула H_2 на том притиску и т-ри од $50\,^\circ C$

$p = 1\ \mu Pa$

$\nu = ?$ молекула H_2

$T = 323{,}15\ K$

$\sigma_{H_2} = 0{,}27\ nm^2$

$\frac{m}{M} = \frac{N}{N_A}$ $\quad m = 3{,}321 \cdot 10^{-24}\ g$

$m = 3{,}321 \cdot 10^{-27}\ kg$

$\nu = \sqrt{2}\,\langle v \rangle \cdot \sigma \cdot \frac{p}{kT}$

$\nu = 1{,}41 \cdot 1849{,}64\ \frac{m}{s} \cdot 0{,}27 \cdot 10^{-18}\ m^2 \cdot \frac{10^{-6}\ Pa}{1{,}38 \cdot 10^{-23}\ J \cdot 323{,}15}$

$\nu = 0{,}16\ \frac{1}{s}$

$\langle v \rangle = \sqrt{\frac{8kT}{\pi m}} = 1849{,}64\ \frac{m}{s}$

5.46 Суд $V = 10^3\ cm^3$ на $T = 25\,^\circ C$ налази се 10^{20} молекула Cl_2 и 10^{23} молекула H_2. Одредити укупан број судара између молекула хлора и водоника по јединици времена и запремине.

$r_{Cl_2} = 0{,}544\ nm$ $\quad T = 298{,}15\ K$

$r_{H_2} = 0{,}293\ nm$

$\rightarrow$ укупан бр. судара између молекула Cl_2 и H_2 износи:

$\nu'_{AB} = r^2 \pi \cdot n_A \cdot n_B \cdot \langle v_r \rangle_{AB}$

$\nu' = (r_{Cl_2} + r_{H_2})^2 \pi \cdot \frac{N_{Cl_2}}{V} \cdot \frac{N_{H_2}}{V} \cdot \sqrt{\frac{8kT}{\pi \mu}}$

$\nu' = 0{,}7 \cdot 10^{-18}\ m^2 \cdot 3{,}14 \cdot \frac{10^{20} \cdot 10^{23}}{1\ 10^{-10}\ m^3} \cdot 1801{,}5\ \frac{m}{s}$

$\nu' = 4 \cdot 10^{38}\ \frac{1}{s \cdot m^3}$

$\mu = \frac{m_{Cl_2} \cdot m_{H_2}}{(m_{Cl_2} + m_{H_2})} = 3{,}23 \cdot 10^{-27}\ kg$

$\frac{m_{Cl_2}}{M_{Cl_2}} = \frac{N_{Cl_2}}{N_A}$ $\quad m_{Cl_2} = 1{,}179 \cdot 10^{-22}\ g$

$m_{Cl_2} = 1{,}179 \cdot 10^{-25}\ kg$

$\frac{m_{H_2}}{M_{H_2}} = \frac{N_{H_2}}{N_A}$ $\quad m_{H_2} = 3{,}321 \cdot 10^{-27}\ kg$

5.47 $\rightarrow$ Израчунати највероватнију кинетичку енергију и варијансу расподеле кинетичке енергије молекула.

$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot v^2 \cdot e^{-\frac{mv^2}{2kT}}$

$\rightarrow$ прелазимо из једне расподеле у другу $f(v) \rightarrow f(\epsilon)$ на бази:

$f(\epsilon)\,d\epsilon = f(v)\,dv$ $\quad f(\epsilon) = f(v)\,\frac{dv}{d\epsilon}$

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$$f(\epsilon) = f(v)\,\frac{dv}{d\epsilon} \quad \Rightarrow \quad \epsilon = \frac{mv^2}{2} \qquad v = \sqrt{\frac{2\epsilon}{m}} \qquad \frac{\partial v}{\partial \epsilon} = \frac{1}{2\sqrt{\frac{2\epsilon}{m}}} \cdot \frac{2}{m} = \frac{1}{m}\sqrt{\frac{m}{2\epsilon}}$$

↳ Максвелова расподела

$$f(\epsilon) = 4\pi \cdot \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot v^2 \cdot e^{-\frac{mv^2}{2kT}} \cdot \frac{1}{m} \cdot \sqrt{\frac{m}{2\epsilon}}$$

$$f(\epsilon) = 4\pi \cdot \frac{m^{\frac{3}{2}}}{2^{\frac{3}{2}} \cdot \pi^{\frac{3}{2}} \cdot (kT)^{\frac{3}{2}}} \cdot \frac{2\epsilon}{m} \cdot e^{-\frac{m 2\epsilon}{2kT m}} \cdot m^{-1} \cdot \frac{m^{\frac{1}{2}}}{2^{\frac{1}{2}} \cdot \epsilon^{\frac{1}{2}}}$$

$$f(\epsilon) = 2^2 \cdot 2^{-\frac{3}{2}} \cdot 2 \cdot 2^{-\frac{1}{2}} \cdot \pi \cdot \pi^{-\frac{3}{2}} \cdot \frac{1}{(kT)^{\frac{3}{2}}} \cdot m^{\frac{3}{2}} \cdot m^{-1} \cdot m^{\frac{1}{2}} \cdot m^{-1} \cdot \epsilon \cdot \epsilon^{-\frac{1}{2}} \cdot e^{-\frac{\epsilon}{kT}} =$$

$$= 2 \cdot \pi^{-\frac{1}{2}} \cdot \frac{1}{(kT)^{\frac{3}{2}}} \cdot m^0 \cdot \epsilon^{\frac{1}{2}} \cdot e^{-\frac{\epsilon}{kT}}$$

$$f(\epsilon) = \frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}} \cdot \sqrt{\epsilon} \cdot e^{-\frac{\epsilon}{kT}}$$

→ Највероватнија енергија се добија из услова: $\frac{\partial f(\epsilon)}{\partial \epsilon} = 0$

$$\frac{\partial f(\epsilon)}{\partial \epsilon} = \frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}} \cdot \left(\frac{1}{2\sqrt{\epsilon}} \cdot e^{-\frac{\epsilon}{kT}} + \sqrt{\epsilon} \cdot e^{-\frac{\epsilon}{kT}} \cdot \left(-\frac{1}{kT}\right)\right)$$

$$= \frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}} \left(\frac{1}{2\sqrt{\epsilon}} \cdot e^{-\frac{\epsilon}{kT}} - \frac{\sqrt{\epsilon}}{kT} \cdot e^{-\frac{\epsilon}{kT}}\right) =$$

$$= \underbrace{\frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}}}_{\neq 0} \cdot e^{-\frac{\epsilon}{kT}} \underbrace{\left(\frac{1}{2\sqrt{\epsilon}} - \frac{\sqrt{\epsilon}}{kT}\right)}_{0} = 0$$

$$\frac{1}{2\sqrt{\epsilon_{nv}}} - \frac{\sqrt{\epsilon_{nv}}}{kT} = 0 \quad \Rightarrow \quad \frac{\sqrt{\epsilon_{nv}}}{kT} = \frac{1}{2\sqrt{\epsilon_{nv}}} \quad \Rightarrow \quad 2 \cdot \epsilon_{nv} = kT \quad = \quad \epsilon_{nv} = \frac{kT}{2}$$

→ Варијанса кинетичке енергије је једнака:

$$\langle (\delta\epsilon)^2 \rangle = \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2$$

$$\langle \epsilon^n \rangle = \int_0^{+\infty} \epsilon^n \cdot f(\epsilon) \cdot d\epsilon \qquad \langle \epsilon \rangle = \int_0^{+\infty} \epsilon \cdot f(\epsilon) \cdot d\epsilon$$

$$\langle \epsilon^2 \rangle = \int_0^{+\infty} \epsilon^2 \cdot f(\epsilon) \cdot d\epsilon$$

$$\langle \epsilon^2 \rangle = \int_0^{+\infty} \epsilon^2 \cdot \frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}} \cdot \sqrt{\epsilon} \cdot e^{-\frac{\epsilon}{kT}} d\epsilon = \frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}} \cdot \int_0^{+\infty} \epsilon^{\frac{5}{2}} \cdot e^{-\frac{\epsilon}{kT}} d\epsilon =$$

смена: $\frac{\epsilon}{kT} = t \qquad dt = \frac{1}{kT} d\epsilon$

$\epsilon = tkT$

$$= \frac{2}{\sqrt{\pi} \cdot (kT)^{\frac{3}{2}}} \cdot \int_0^{\infty} (kT)^{\frac{5}{2}} \cdot t^{\frac{5}{2}} \cdot e^{-t} \cdot kT\, dt =$$

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$$= \frac{2}{\sqrt{\pi}} \cdot (kT)^{-\frac{3}{2}} \cdot (kT)^{\frac{7}{2}} \cdot \underbrace{\int_0^{\infty} t^{\frac{5}{2}} \cdot e^{-t}\, dt}_{\Gamma\left(\frac{7}{2}\right)} = \frac{2}{\sqrt{\pi}} \cdot (kT)^2 \cdot \frac{15}{4}\sqrt{\pi} = \frac{15}{4}(kT)^2$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

nowa: $\frac{7}{2} - 1 = \frac{5}{2}$

$$\langle \epsilon \rangle = \int_0^{+\infty} \frac{2\epsilon}{\sqrt{\pi}\cdot(kT)^{\frac{3}{2}}} \cdot \sqrt{\epsilon} \cdot e^{-\frac{\epsilon}{kT}}\, d\epsilon =$$

$$= \frac{2}{\sqrt{\pi}\cdot(kT)^{\frac{3}{2}}} \cdot \int_0^{+\infty} \epsilon^{\frac{3}{2}} \cdot e^{-\frac{\epsilon}{kT}}\, d\epsilon =$$

ZMIENNA:
$\frac{\epsilon}{kT} = t \qquad d\epsilon = kT\, dt$
$\epsilon = kT \cdot t$

$$= \frac{2}{\sqrt{\pi}\cdot(kT)^{\frac{3}{2}}} \cdot \int_0^{+\infty} (kT)^{\frac{3}{2}} \cdot t^{\frac{3}{2}} \cdot e^{-t} \cdot k \cdot T \cdot dt =$$

$$= \frac{2}{\sqrt{\pi}} \cdot (kT)^{-\frac{3}{2}} \cdot (kT)^{\frac{5}{2}} \cdot \underbrace{\int_0^{+\infty} t^{\frac{3}{2}} \cdot e^{-t}\, dt}_{\Gamma\left(\frac{5}{2}\right)} = \frac{2}{\sqrt{\pi}} \cdot (kT) \cdot \frac{3}{2}\sqrt{\pi} = \frac{3}{2}kT$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\langle \epsilon \rangle^2 = \frac{9}{4}(kT)^2$$

$$\langle (\delta\epsilon)^2 \rangle = \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 = \frac{15}{4}(kT)^2 - \frac{9}{4}(kT)^2 = \frac{6}{4}(kT)^2 = \frac{3}{2}(kT)^2$$

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