ЕЛЕМЕНТАРНЕ ЧЕСТИЦЕ

Бозони
$s = 0, 1, 2, \dots$
$h\nu, H, d, \dots$

Фермиони
$s = \frac{1}{2}, \frac{3}{2}, \dots$
$e^-, e^+, p^+, n, \dots$

$$n_j = \begin{cases} 0, 1 & \phi \\ 0, 1, 2, \dots & б \end{cases}$$

$$\begin{pmatrix}\text{број}\\ \text{честица}\end{pmatrix} \ll \begin{pmatrix}\text{број}\\ \text{доступних}\\ \text{стања}\end{pmatrix} \Rightarrow \Gamma_\phi = \Gamma_б \approx \frac{\text{број доступних стања}}{N!}$$

$$\Xi = \sum_{N=0}^{\infty} \sum_i e^{-\beta(E_i - \mu N)} \Rightarrow \Xi_{\substack{ф\\ба}} = \prod_k (1 \pm \lambda e^{-\beta \varepsilon_k})^{\pm 1} \qquad e^{\beta\mu} = \lambda$$

$$\langle N \rangle = \sum_k \langle n_k \rangle = \left(\frac{\partial \ln \Xi}{\partial \beta \mu}\right)_{\beta, V} = \lambda \left(\frac{\partial \ln \Xi}{\partial \lambda}\right)_{\beta, V} = \lambda \frac{\partial}{\partial \lambda}\left[\pm \sum_k \ln(1 \pm \lambda e^{-\beta \varepsilon_k})\right]$$

$$= \pm \lambda \sum_k \frac{\pm e^{-\beta \varepsilon_k}}{1 \pm \lambda e^{-\beta \varepsilon_k}} = \sum_k \underbrace{\frac{\lambda e^{-\beta \varepsilon_k}}{1 \pm \lambda e^{-\beta \varepsilon_k}}}_{\langle n_k \rangle}$$

$$\boxed{\langle n_k \rangle = \frac{\lambda e^{-\beta \varepsilon_k}}{1 \pm \lambda e^{-\beta \varepsilon_k}} = \frac{1}{e^{\beta(\varepsilon_k - \mu)} \pm 1}}$$

Густина стања за бозоне и фермионе

$$\vec{k} = \frac{\pi}{L}(n_x \vec{i} + n_y \vec{j} + n_z \vec{k}) \quad ; \quad n_x, n_y, n_z = 1, 2, 3, \dots$$

$$\psi(\vec{r}, t) = A e^{i(\vec{k}\vec{r} - \omega t + \varphi)}$$

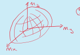
$$v_p = \frac{\omega}{|\vec{k}|} \qquad v_g = \frac{d\omega}{dk}$$

$$\vec{S} = \vec{E} \times \vec{H} \qquad \vec{S} \parallel \vec{k}$$

$$V = L^3$$

$$\vec{k} \cdot \vec{k} = k^2 = \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$$

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{Lk}{\pi}\right)^2$$



$$\begin{pmatrix}\text{број}\\ \text{стања}\end{pmatrix} = \frac{1}{8}\begin{pmatrix}\text{запремине}\\ \text{сфере}\end{pmatrix}$$

$$\Phi(k) = \frac{1}{8} \cdot \frac{4}{3}\pi \frac{\overset{V}{L^3} k^3}{\pi^3}$$

$$\Phi(k) = \frac{k^3 V}{6\pi^2}$$

$$g(k) = \frac{d\Phi(k)}{dk} = \frac{k^2 V}{2\pi^2}$$

$$g(k) \to g(\varepsilon)$$

$$g(k)\,dk = g(\varepsilon)\,d\varepsilon$$

$$g(\varepsilon) = \frac{k^2 V}{2\pi^2}\frac{dk}{d\varepsilon}$$

„$k \; \hbar \; h$"
$\downarrow \; \downarrow \; \downarrow$
$K \; h \; X$

Бозони и фермиони са масом

1) $\varepsilon = \frac{p^2}{2m}$

$$h = p \cdot \lambda \Rightarrow p = \frac{h}{\lambda} \cdot \frac{2\pi}{2\pi} = \hbar k$$

$$\varepsilon = \frac{\hbar^2 k^2}{2m} \Rightarrow k = \frac{\sqrt{2m\varepsilon}}{\hbar} \Big/ \frac{d}{d\varepsilon}$$

$$\frac{dk}{d\varepsilon} = \frac{1}{2\hbar}\sqrt{\frac{2m}{\varepsilon}}$$

$$g(\varepsilon) = \frac{k^2 V}{2\pi^2}\frac{dk}{d\varepsilon} = \frac{V}{2\pi^2}\frac{2m\varepsilon}{\hbar^2}\frac{1}{2\hbar}\sqrt{\frac{2m}{\varepsilon}}$$

$$\boxed{\begin{gathered} g(\varepsilon) = 2\pi V \left(\frac{2m}{h}\right)^{\frac{3}{2}} \sqrt{\varepsilon} \\ g(\varepsilon) \sim \varepsilon^{\frac{1}{2}} \end{gathered}}$$

Бозони без масе

2) $\varepsilon = h\nu = \hbar k c \Big/ \frac{d}{d\varepsilon}$

$$g(\varepsilon) = 4\pi V \frac{1}{(hc)^3}\varepsilon^2$$

$$g(\varepsilon) \sim \varepsilon^2$$

Идеалан фотонски гас

$$g(\varepsilon) = 2 \cdot \left[4\pi V \frac{1}{(hc)^3}\varepsilon^2\right]$$

$$\langle N \rangle = \sum_k \langle n_k \rangle = \sum_k \frac{\lambda e^{-\beta \varepsilon_k}}{1 - \lambda e^{-\beta \varepsilon_k}} \quad (1)$$

$$\langle E \rangle = \sum_k \varepsilon_k \langle n_k \rangle = \sum_k \frac{\varepsilon_k \lambda e^{-\beta \varepsilon_k}}{1 - \lambda e^{-\beta \varepsilon_k}} \quad (2)$$

$$\Omega = -PV = -kT \ln \Xi = -kT \sum_k \ln(1 - \lambda e^{-\beta \varepsilon_k})^{-1}$$

$$P = -\frac{kT}{V}\sum_k \ln(1 - \lambda e^{-\beta \varepsilon_k}) \quad (3)$$

$$\sum_k f(\varepsilon_k) \to \int g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$\varepsilon = \omega \hbar \rightsquigarrow g(\omega) = \frac{V\omega^2}{\pi^2 c^3}$$

$$g(\varepsilon) \to g(\omega)$$

$$g(\varepsilon)\,d\varepsilon = g(\omega)\,d\omega$$

$$g(\omega) = g(\varepsilon)\underset{\hbar}{\frac{d\varepsilon}{d\omega}}$$

$$(1) \to \langle N \rangle = \frac{V}{\pi^2 c^3}\int_0^\infty \frac{\omega^2 e^{-\beta\omega\hbar}}{1 - e^{-\beta\omega\hbar}}\,d\omega \quad (4)$$

$$(2) \to \langle E \rangle = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3 e^{-\beta\omega\hbar}}{1 - e^{-\beta\omega\hbar}}\,d\omega \quad (5)$$

$$(3) \to P = -\frac{kT}{\pi^2 c^3}\int_0^\infty \omega^2 \ln(1 - e^{-\beta\omega\hbar})\,d\omega \quad (6)$$

(4) и (5):
$$\frac{e^{-\beta\omega\hbar}}{1 - e^{-\beta\omega\hbar}} = e^{-\beta\omega\hbar}\frac{1}{1 - e^{-\beta\omega\hbar}} = e^{-\beta\omega\hbar}\sum_{n=0}^{\infty}(e^{-\beta\omega\hbar})^n = \sum_{n=1}^{\infty} e^{-\beta n\omega\hbar}$$

(6):
$$\ln(1 - e^{-\beta\omega\hbar}) = -\sum_{n=1}^{\infty}\frac{e^{-\beta\omega\hbar n}}{n}$$

$$(5) \to \langle E \rangle = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \omega^3 \sum_{n=1}^{\infty} e^{-\beta n\omega\hbar}\,d\omega =$$

$$\beta n\omega\hbar = x \Rightarrow \omega = \frac{x}{\beta n\hbar} \qquad d\omega = \frac{dx}{\beta n\hbar}$$

$$= \frac{V\hbar}{\pi^2 c^3}\int_0^\infty \frac{x^3}{(\beta n\hbar)^3}\sum_{n=1}^{\infty} e^{-x}\frac{dx}{\beta n\hbar} =$$

$$= \frac{V\hbar}{\pi^2 c^3}\frac{1}{(\beta\hbar)^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\underbrace{\int_0^\infty x^3 e^{-x}\,dx}_{\Gamma(4) = 3! = 6} =$$

$$\langle E \rangle = \frac{6V(kT)^3}{\pi^2(c\hbar)^3} \cdot kT\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Зета функција

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x} \qquad \zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t - 1}\,dt$$

$$\zeta(1) \to \infty \qquad \zeta(2) = \frac{\pi^2}{6} \qquad \lim_{x \to 1}(x-1)\zeta(x) = 1$$

$$\zeta(3) \to 1{,}20205\dots \qquad \zeta(4) = \frac{\pi^4}{90}$$

Општи случај зависности PV~E

$$g(\varepsilon) = A\varepsilon^B \qquad A, B = const$$

$$PV = f(A, B)\langle E \rangle$$

$$\langle E \rangle = \sum_i \langle n_i \rangle \varepsilon_i = \sum_i \frac{\varepsilon_i \lambda e^{-\beta \varepsilon_i}}{1 \pm \lambda e^{-\beta \varepsilon_i}} \longrightarrow \int_0^\infty A\varepsilon^B \frac{\varepsilon\lambda e^{-\beta\varepsilon}}{1 \pm \lambda e^{-\beta\varepsilon}}\,d\varepsilon =$$

$$= A\int_0^\infty \frac{\varepsilon^{B+1}\lambda e^{-\beta\varepsilon}}{1 \pm \lambda e^{-\beta\varepsilon}}\,d\varepsilon$$

$$u = \varepsilon^{B+1} \qquad du = (B+1)\varepsilon^B$$

$$d\vartheta = \frac{\lambda e^{-\beta\varepsilon}\,d\varepsilon}{1 \pm \lambda e^{-\beta\varepsilon}} \Rightarrow \vartheta = \int \frac{\lambda e^{-\beta\varepsilon}}{1 \pm \lambda e^{-\beta\varepsilon}}\,d\varepsilon \quad \left[\begin{gathered} 1 \pm \lambda e^{-\beta\varepsilon} = t \\ \pm\lambda e^{-\beta\varepsilon}(-\beta)\,d\varepsilon = dt \\ \lambda e^{-\beta\varepsilon}\,d\varepsilon = \mp\frac{dt}{\beta} \end{gathered}\right]$$

$$\vartheta = \int \frac{\mp\frac{dt}{\beta}}{t} = \mp\frac{1}{\beta}\int\frac{dt}{t} = \mp kT\ln(1 \pm \lambda e^{-\beta\varepsilon})$$

$$\langle E \rangle = A\left[\mp kT\varepsilon^{B+1}\cancelto{0}{\ln(1 \pm \lambda e^{-\beta\varepsilon})\Big|_0^\infty} \pm kT(B+1)\int_0^\infty \varepsilon^B \ln(1 \pm \lambda e^{-\beta\varepsilon})\,d\varepsilon\right]$$

$$\langle E \rangle = \underline{\pm kT A}(B+1)\underline{\int_0^\infty \varepsilon^B \ln(1 \pm \lambda e^{-\beta\varepsilon})\,d\varepsilon} \quad (1)$$

$$\Omega = -PV = -kT\ln\Xi \qquad \Xi = \prod_i (1 \pm e^{-\beta\varepsilon_i})^{\pm 1}$$

$$PV = \pm kT\sum_i \ln(1 \pm e^{-\beta\varepsilon_i}) \longrightarrow \pm kT\int A\varepsilon^B \ln(1 \pm e^{-\beta\varepsilon})\,d\varepsilon \quad (2)$$

$$\langle E \rangle = (B+1)PV \rightsquigarrow \boxed{PV = \frac{1}{B+1}\langle E \rangle}$$

Бозони и фермиони са масом

1)
$$g(\varepsilon) = A\varepsilon^B$$
$$g(\varepsilon) \sim \varepsilon^{\frac{1}{2}}$$
$$PV = \frac{1}{\frac{1}{2} + 1}\langle E \rangle$$
$$PV = \frac{2}{3}\langle E \rangle$$

Бозони без масе

2)
$$g(\varepsilon) \sim \varepsilon^2$$
$$PV = \frac{1}{3}\langle E \rangle$$

Слабодегенерисани бозонски гас

$$\lambda = e^{\beta\mu} \approx 1 \Rightarrow \mu \approx 0$$

$$\langle E \rangle = A + B\rho + C\rho^2 + \dots \qquad \rho = \frac{\langle N \rangle}{V}$$

$$P = A' + B'\rho + C'\rho^2 + \dots$$

$$g(\varepsilon) = 4\pi V\frac{\varepsilon^2}{(hc)^3}$$

$$\frac{1}{1 \pm x} = \sum_{n=0}^{\infty}(\mp 1)^n x^n = \sum_{n=1}^{\infty}(\mp 1)^{n-1}x^{n-1}$$

$$\ln(1 \pm x) = \sum_{n=1}^{\infty}(-1)^{\substack{n+1\\2n+1}}\frac{x^n}{n}$$

$$\langle N \rangle = \int_0^\infty g(\varepsilon)\frac{\lambda e^{-\beta\varepsilon}}{1 \pm \lambda e^{-\beta\varepsilon}}\,d\varepsilon$$

$$\langle E \rangle = \int_0^\infty g(\varepsilon)\frac{\varepsilon\lambda e^{-\beta\varepsilon}}{1 \pm \lambda e^{-\beta\varepsilon}}\,d\varepsilon$$

$$P = \pm\frac{kT}{V}\int_0^\infty g(\varepsilon)\ln(1 \pm \lambda e^{-\beta\varepsilon})\,d\varepsilon$$

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$$\langle N \rangle = \frac{4\pi V}{(hc)^3}\int_0^\infty \lambda e^{-\beta\varepsilon}\sum_{n=1}^{\infty}(\lambda e^{-\beta\varepsilon})^{n-1}\,d\varepsilon =$$

$$= \frac{4\pi V}{(hc)^3}\sum_{n=1}^{\infty}\lambda^n\int_0^\infty e^{-\beta n\varepsilon}\,d\varepsilon \Rightarrow \qquad \beta n\varepsilon = x \; \dots \Rightarrow \Gamma(3) = 2$$

$$\Rightarrow \langle N \rangle = 8\pi V\left(\frac{kT}{hc}\right)^3\sum_{n=1}^{\infty}\frac{\lambda^n}{n^3}$$

$$\boxed{\frac{\langle N \rangle}{V} = \rho = \pi\left(\frac{2kT}{hc}\right)^3\sum_{n=1}^{\infty}\frac{\lambda^n}{n^3}}$$

$$\rho = \pi\left(\frac{2kT}{hc}\right)^3\left[\lambda + \frac{1}{8}\lambda^2 + \frac{1}{27}\lambda^3 + \dots\right]$$

$$\lambda = a_1\rho + a_2\rho^2 + \dots$$

$$\rho = \pi\left(\frac{2kT}{hc}\right)^3\left[a_1\rho + a_2\rho^2 + \dots + \frac{1}{8}(a_1\rho + a_2\rho^2 + \dots)^2 + \dots\right]$$

$$\underbrace{\frac{1}{\pi}\left(\frac{hc}{2kT}\right)^3}\cdot\rho + \underbrace{0}\cdot\rho^2 + \dots = \underbrace{a_1}\rho + \underbrace{\left(a_2 + \frac{1}{8}a_1^2\right)}\rho^2 + \dots$$

$$a_1 = \frac{1}{\pi}\left(\frac{hc}{2kT}\right)^3 \qquad a_2 + \frac{1}{8}a_1^2 = 0$$

$$a_2 = -\frac{1}{8\pi^2}\left(\frac{hc}{2kT}\right)^6$$

$$P = -\frac{4\pi kT}{(hc)^3}\int_0^\infty \varepsilon^2\sum_{n=1}^{\infty} - \frac{(\lambda e^{-\beta\varepsilon})^n}{n}\,d\varepsilon$$

$$\boxed{P = kT\left(\frac{2kT}{hc}\right)^3\pi\sum_{n=1}^{\infty}\frac{\lambda^n}{n^4}}$$

$$\boxed{\lambda = \frac{1}{\pi}\left(\frac{hc}{2kT}\right)^3\rho - \frac{1}{8\pi^2}\left(\frac{hc}{2kT}\right)^6\rho^2 + \dots}$$

$$P = kT\left(\frac{2kT}{hc}\right)^3\pi\left[\frac{\lambda^1}{1^4} + \frac{\lambda^2}{2^4} + \dots\right]$$

$$P = kT\left(\frac{2kT}{hc}\right)^3\pi\left[\frac{1}{\pi}\left(\frac{hc}{2kT}\right)^3\rho - \frac{1}{8\pi^2}\left(\frac{hc}{2kT}\right)^6\rho^2 + \dots + \frac{1}{16}\frac{1}{\pi^2}\left(\frac{hc}{2kT}\right)^6\rho^2 + \dots\right]$$

$$P = \frac{\langle N \rangle kT}{V}\left[1 - \frac{1}{16\pi}\left(\frac{hc}{2kT}\right)^3\rho + \dots\right]$$

$$\langle E \rangle = 3\langle N \rangle kT\left[1 + \dots\right]$$